Stationary distribution of finite-size systems with absorbing states

Tânia Tomé and Mário J. de Oliveira
Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, 05315-970 São Paulo, São Paulo, Brazil

(Received 30 May 2005; published 26 August 2005)

We introduce a procedure that allows us to obtain nontrivial stationary distributions of finite-size models with absorbing states. Two models are studied: the contact process and the sandpile model with height restriction. To avoid the permanence of the system in the absorbing state we create a small perturbation that drives the system out of the absorbing state. In the former model a particle is created, in the latter an active site is created. The stationary distributions around the critical point are analyzed by the use of finite-size scaling.

DOI: 10.1103/PhysRevE.72.026130 PACS number(s): 02.50.Ga, 05.10.−a, 05.70.Ln

I. INTRODUCTION

The thermodynamic properties of a given system is obtained by performing the thermodynamic limit in which the densities are kept constant while the size of the system grows without limits. If we are using Monte Carlo simulations, in which case the system is always finite, it suffices to study systems with sizes larger than the correlation length. If our purpose is the study of the critical region, we should keep in mind that the correlation length diverges at the critical point and we cannot anymore study systems with sizes larger than the correlation length. However, this problem can be overcome by the help of the finite-size scaling hypothesis. That is, numerical simulations are performed on finite systems of several sizes. If the systems are big enough so that they are in the finite-size scaling region then one can extract the thermodynamic limit properties including the critical behavior.

The study of finite systems on the other hand poses a particular problem which concerns models with one absorbing state such as the contact process [1–3] or models with infinitely many absorbing states [4–11]. Here we analyze only two representative models: the usual contact process and the sandpile model with height restriction [11]. In the thermodynamic limit, these systems exhibit two phases: an active phase where the order parameter ρ is nonzero (supercritical regime) and a nonactive phase in which $\rho=0$ (subcritical regime). However, if the system is finite the nonequilibrium dynamics leads the system to fall into the absorbing state and ρ will be zero even in the supercritical regime. Of course, this may not happen in an actual simulation of the active state if the number of Monte Carlo steps is not sufficiently large. In any case, to avoid the permanence of the system in the absorbing state, we use the following procedure: (a) in the contact process we create a particle (or, equivalently, we forbid the last particle to be annihilated), and (b) in the sandpile model with height restriction an active site is created by moving a particle on the top of another one. Concerning the properties of the system in the thermodynamic limit, these perturbations are found to be irrelevant, as shown by our numerical results.

By the above procedure, we are eliminating, strictly speaking, the absorbing state so that now ρ is always nonzero. This is not a problem if we are in the supercritical regime in which the order parameter is in fact nonzero. In the subcritical regime, although ρ is nonzero it will decrease and eventually vanish in the thermodynamic limit. One advantage is that now ρ will be a continuous and differentiable function of the external parameter, being treatable by a finite-size scaling analysis.

Here we study numerically the contact process and the sandpile model with height restriction. The procedure explained above allows us to determine the stationary distribution at the critical point and in the subcritical regime as well. We also determine the critical point and the critical exponents for both models. The exponents and the nonuniversal parameters are found to be in agreement with results obtained in the standard manner.

It is worth mentioning that there are other ways of generating the stationary state for finite-size systems. de Oliveira and Dickman [12] define a quasistationary distribution by studying the ordinary model as function of time. Ziff and Brozilow [13] and Tomé and de Oliveira [14] used conserved versions of the original models in which the absorbing state has been eliminated.

II. FINITE-SIZE SCALING

Let us consider a finite system of linear size L and let ε be the deviation of the external parameter from its critical value. The quantities of interest are $\rho=\langle x\rangle$ and $q=\langle x^2\rangle$ where x is a stochastic variable associated to the order parameter. The quantities ρ and q can be obtained from the stationary probability density $P(x)$ of the stochastic variable x. In order to set up the finite-size scaling relations for the quantities of interest we will assume the following scaling

$$P(L,\varepsilon,x)dx = \widetilde{P}(\varepsilon L^{1/\nu_\perp}, xL^{\beta/\nu_\perp})L^{\beta/\nu_\perp}dx, \tag{1}$$

where β and $\nu_\perp$ are the critical exponents associated with the order parameter and the spatial correlation length, and $\widetilde{P}(y,z)$ is a universal probability density function. From Eq. (1), we obtain scaling relations for the quantities ρ and q which are

$$\rho(L,\varepsilon) = L^{-\beta/\nu_\perp}\widetilde{\rho}(\varepsilon L^{1/\nu_\perp}) \tag{2}$$

and

$$q(L,\varepsilon) = L^{-2\beta/\nu_\perp}\widetilde{q}(\varepsilon L^{1/\nu_\perp}) \tag{3}$$

where $\widetilde{\rho}(y)$ and $\widetilde{q}(y)$ are universal functions.

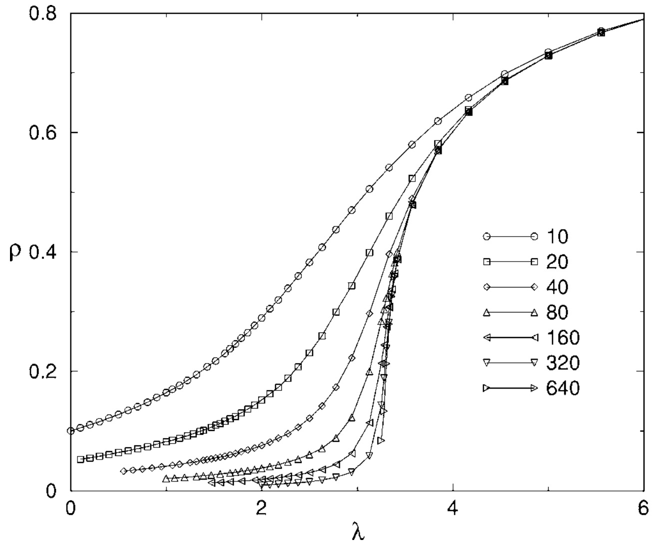

FIG. 1. Density of particles ρ for the contact process as a function of the creation rate λ for several values of the system size L.

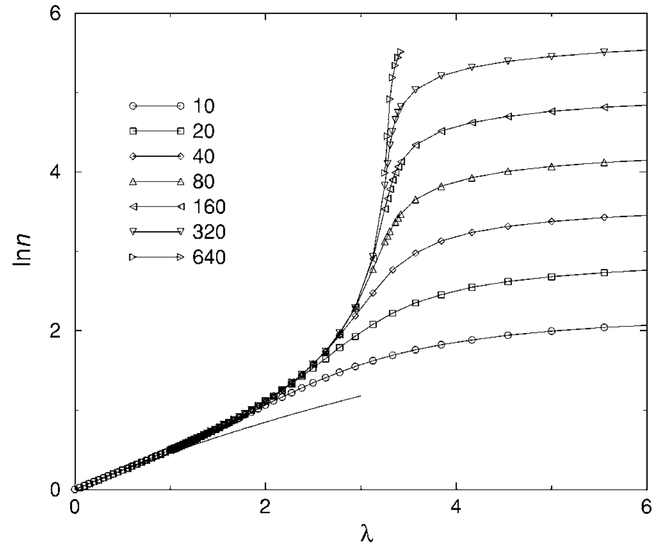

FIG. 2. Number of particles $n=L\rho$ for the contact process as a function of the creation rate λ for several values of the system size L. The continuous line represents the expansion of n up to λ^2 as given by Eq. (20).

Defining the reduced second-order cumulant u by

$$u=\frac{q-\rho^2}{\rho^2} \tag{4}$$

we get the following scaling relation:

$$u(L,\varepsilon)=\tilde{u}(\varepsilon L^{1/\nu_\perp}) \tag{5}$$

where $\tilde{u}(y)$ is a universal function. This reduced second-order cumulant is useful to determine the critical point since at the critical point $u(L,0)$ will be independent of L.

In the supercritical regime, and in the limit $L\to\infty$, the order parameter behaves as

$$\rho \sim \varepsilon^{\beta}. \tag{6}$$

In the subcritical regime, the relevant quantity is the average number of particle, in the contact process, or the number of active particles, in the sandpile model with height restriction, given by $n=L^d\rho$. In the limit $L\to\infty$ this quantity behaves as

$$n \sim |\varepsilon|^{-\sigma}, \quad \sigma=d\nu_\perp-\beta, \tag{7}$$

which is a result obtained by combining the scaling relation (2) with the assumption that ρ decays as $1/L^d$ in the subcritical regime.

III. THE CONTACT PROCESS

A. Finite system

We have simulated the one-dimensional contact process with creation rate equal to λ and annihilation rate equal to unity, with L sites and periodic boundary conditions. Each site of the lattice can be either empty or occupied by just one particle. The simulation is performed as follows. At each time step, a site is chosen at random, with probability $1/L$. (a) If the site is empty then one looks at one of its neighboring sites with equal probability. If the chosen neighboring site is occupied then a particle is created at the chosen site with probability $\lambda/(1+\lambda)$. (b) If the site is occupied by a particle then the particle is annihilated with probability $1/(1+\lambda)$. If the particle is the last one in the lattice it is not annihilated. At each time step the time is increased by an interval equal to $1/L$.

We have calculated in the stationary state the density of particles ρ and the reduced second-order cumulant u for several values of L. Figure 1 shows ρ as a function of λ for several values of L. As expected the density ρ is always nonzero. In the supercritical regime it accumulates at a certain value as $L\to\infty$, but in the subcritical regime it decreases and vanishes, in this limit. The same behavior can be seen in Fig. 2 where $n=L\rho$ is plotted as a function of λ. In the subcritical regime n accumulates at a certain value and in the supercritical it increases without bounds. Figure 3 shows the reduced second-order cumulant u as a function of λ for several values of L. From the crossing of u, we determine the critical values $\lambda_c=3.298(2)$ and $u_c=0.205(4)$.

From the log-log plot of $\rho(L,0)$ versus L we determine the exponent ratio $\beta/\nu_\perp$ by Eq. (2). The exponent β and $\nu_\perp$ can be obtained separately by the use of Eqs. (6) and (7). The numerical results obtained from the numerical data are in agreement with the accepted values, namely, $\beta=0.277$ and $\nu_\perp=1.097$ for the contact process [2]. We have used these values for the exponent β and $\nu_\perp$ to obtain the data collapse for the density ρ shown in Fig. 4. Although the values of L are not large, a good collapse of the data is obtained.

B. Infinite system

In the subcritical regime the density of particles ρ will vanish in the thermodynamic limit. It is therefore convenient in this case to simulate an infinite system with a finite number of particles. This is performed as follows. At each time step a particle (instead of a site) is chosen at random, with probability $1/X$, where X is the number of particles. (a) With

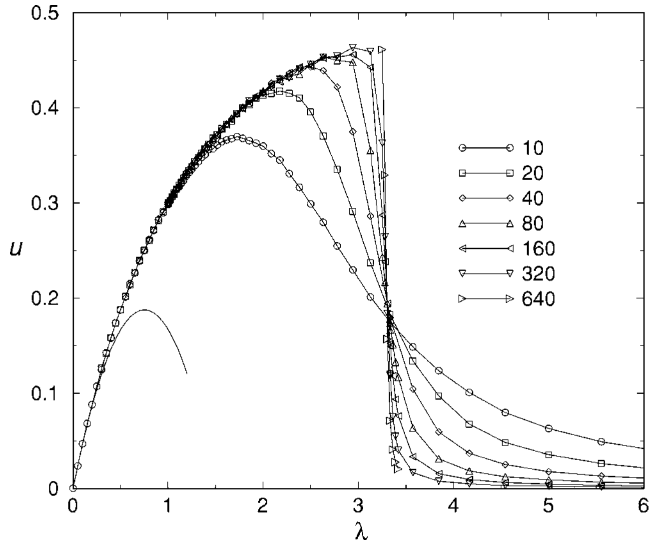

FIG. 3. Reduced second-order cumulant u for the contact process as a function of the creation rate λ for several values of the system size L. The continuous line represents the expansion of u up to λ^2 as given by Eq. (22).

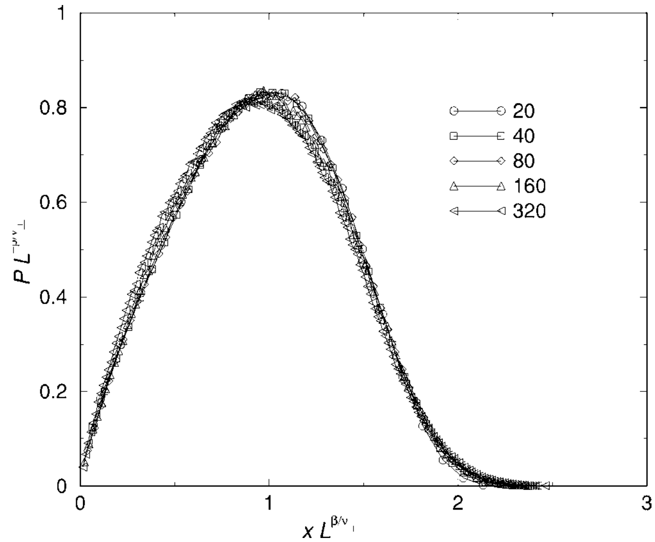

FIG. 4. Data collapse of probability density $P(L,0,x)$ at the critical point obtained for several values of L.

probability $1/(1+\lambda)$, the particle is annihilated, except if it is the last particle in which case nothing is done. (b) With probability $\lambda/(1+\lambda)$, one chooses one of the two nearest neighbor sites of the particle. If the neighboring site is empty then a new particle is created at this site. At each time step the time is increased by an interval equal to $1/X$. In an actual simulation one may use a finite system and check whether the particles being created do not reach the border of the lattice.

We have performed the numerical simulation on an infinite system by starting from a configuration with just one particle. As long as the $\lambda<\lambda_c$ the number of particles remains finite. It does not increase without bounds as would occur for $\lambda \geqslant \lambda_c$. For each value of λ, there is a cluster of particles that becomes a fractal at the critical point [15]. We have determined the average number of particles n for several values of λ. We have also determined the distribution of the size of the cluster of particles. This is done by measuring the distance ℓ between the two particles at the border of the cluster. It is found that the probability distribution of ℓ decays exponentially, for sufficiently large ℓ. From the exponential decay we get a measure of the spatial correlation length ξ. Around the critical point the quantity ξ behaves as

$$\xi \sim |\varepsilon|^{-\nu_\perp}. \tag{8}$$

On the other hand, n and ξ are related to each other by

$$n \sim \xi^{d_F} \tag{9}$$

where d_F is the fractal dimension of the critical cluster. From the relation (7) it follows that $d_F = d - \beta/\nu_\perp$.

Another quantity of interest is the time T between two consecutive passages of the system to a configuration with just one particle. It is found that the probability distribution of the first passage time T decays exponentially, for sufficiently large T. From the exponential decay we get a measure for the time correlation length τ. Around the critical point τ behaves as

$$\tau \sim |\varepsilon|^{-\nu_\parallel}. \tag{10}$$

Therefore, the relation between ξ and τ is given by

$$\tau \sim \xi^{z} \tag{11}$$

where $z = \nu_\parallel / \nu_\perp$ is the dynamical exponent.

From the numerical simulation we have determined the quantities n, ξ and τ for several values of λ in the subcritical regime. The exponents obtained from the numerical data are consistent with those of the directed percolation universality class [2].

C. Regime of small creation rate

In the regime of small creation rate we show that it is possible to obtain the stationary distribution by means of a series expansion in λ. These results can then be compared with the above numerical results. When the creation rate is sufficiently small the number of particles will be small and there are only a few relevant configurations. Let us denote by A the state with just one particle (…0001000…); B the state with two particles in a row (…0011000…l); C the state with three particles in a row (…0011100…); and D the state with two particles separated by a vacant site (…0010100…). From the dynamic rules we can write down the following equations for the stationary probability of these states:

$$\lambda P(A) = 2P(B) + 2P(D), \tag{12}$$

$$\lambda P(B) + 2P(B) = \lambda P(A) + 2P(C), \tag{13}$$

$$\lambda P(C) + 3P(C) = \lambda P(B) + \lambda P(D), \tag{14}$$

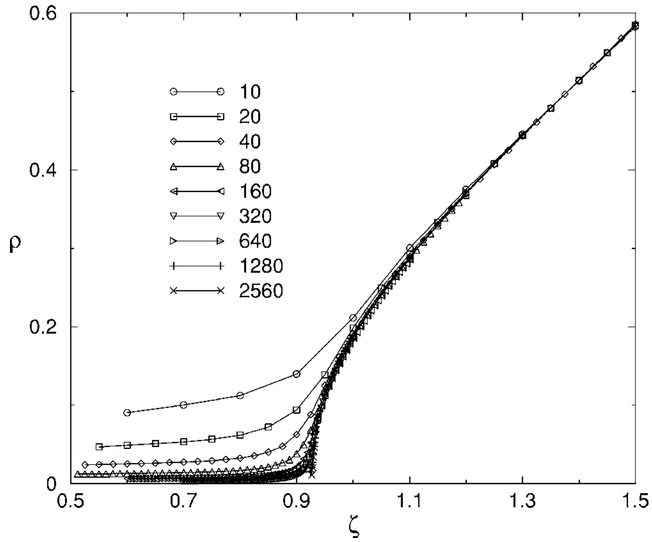

FIG. 5. Density of active sites ρ for the sandpile model with height restriction as a function of the density of particles ζ for several values of the system size L.

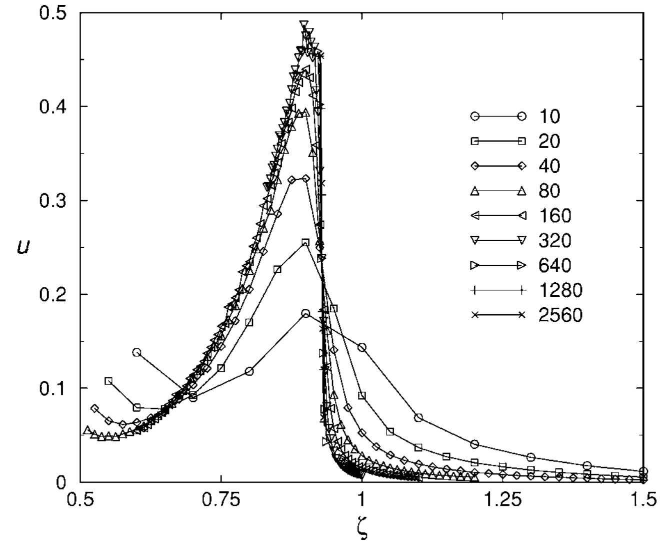

FIG. 6. Reduced second-order cumulant u for the sandpile model with height restriction as a function of the density of particles ζ for several values of the system size L.

$$2\lambda P(D) + 2P(D) = P(C), \tag{15}$$

where we have omitted the terms related to any other configurations. Next we assume that $P(A)$ is of order unity, that $P(B)$ is of order λ, and that $P(C)$ and $P(D)$ are of order λ^2. The solution up to order λ^2 is then

$$P(A) = 1 - \frac{\lambda}{2} + \frac{\lambda^2}{12}, \tag{16}$$

$$P(B) = \frac{\lambda}{2} - \frac{\lambda^2}{3}, \tag{17}$$

$$P(C) = \frac{\lambda^2}{6}, \tag{18}$$

$$P(D) = \frac{\lambda^2}{12}. \tag{19}$$

From these solutions we get the average number of particles

$$n = \langle X \rangle = 1 + \frac{\lambda}{2} + \frac{\lambda^2}{12} + O(\lambda^3) \tag{20}$$

and the second-order cumulant

$$\langle X^2 \rangle - \langle X \rangle^2 = \frac{\lambda}{2} + \frac{\lambda^2}{6} + O(\lambda^3). \tag{21}$$

The reduced second-order cumulant $u = \langle X^2 \rangle / \langle X \rangle^2 - 1$ will be

$$u = \frac{\lambda}{2}\left(1 - \frac{2}{3}\lambda\right) + O(\lambda^3). \tag{22}$$

The result (20) tells us that the particle number is finite which implies that the particle density $\rho = n/L$ will decrease as $1/L$ as expected for the subcritical regime. The decrease in ρ with L can be seen in Fig. 1. The second-order cumulant and u are also finite in the subcritical regime. However, n and the second-order cumulant diverges at the critical point whereas u remains finite as can be inferred from Fig. 3 and from the scaling relations (2) and (3).

IV. SANDPILE WITH HEIGHT RESTRICTION

We have simulated a sandpile model with height restriction corresponding to the independent toppling rules [11], defined as follows. Each site of a lattice can either be empty or occupied by one or two particles. Sites with two particles are said to be active sites. Empty sites and sites with just one particle are inactive sites. At each time step, a site is chosen at random. If it is active then each one of the particles topples, in sequence, to one of the neighboring sites with equal probability. If the neighboring site is occupied by two particles, then the particle does not topple.

When the number of particles, which is a conserved quantity, is smaller than the number of sites then the system may fall in one of the many absorbing configurations. An absorbing configuration is the one having all sites inactive. When the system falls into one of the absorbing configuration then an active site is created by moving a particle on the top of another one.

The order parameter is defined by $\rho = n_a/L$ where n_a is the average number of active sites and L the number of sites in the lattice. The control parameter is the density of particles $\zeta = n/L$ where n is the number of particle, which is a conserved quantity. When $\zeta > 1$ there is always active sites so that the system is in the active state in this regime. When $\zeta \leq 1$ the system might be found in one of the infinity many absorbing state. There is a critical value ζ_c, which is strictly less than the unity, below which the system is found in the nonactive state.

We have calculated the order parameter ρ and the reduced cumulant u for several values of the system size L. Figure 5

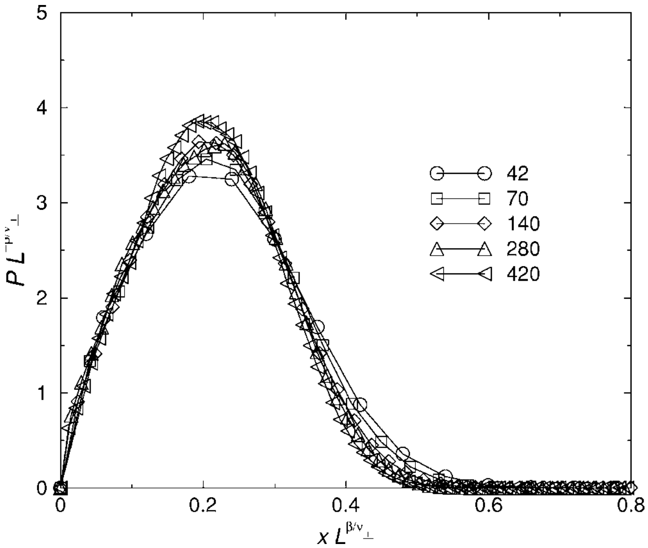

FIG. 7. Data collapse of probability density $P(L,0,x)$ at the critical point obtained for several values of L for the sandpile with height restriction.

shows ρ versus the density ζ for several values of the size of the system L. As expected the order parameter decreases and vanishes with increasing L in the subcritical regime. Figure 6 shows the reduced second-order cumulant as a function of ζ for several values of L. The curves cross each other at the point ζ_c and u_c. From the numerical data we found $\zeta_c = 0.9293(3)$ which is in fair agreement with the result obtained by Dickman *et al.* [11], namely, $\zeta_c = 0.929\,65(3)$.

For the sandpile model it is also possible to get the probability distribution $P(L,\varepsilon,x)$ of the density of the active sites $x = n_a/L$. Figure 7 shows the data collapse of $P(L,0,x)$ at the critical density obtained by using the value $\beta/\nu_\perp = 0.247$ estimated by Dickman *et al.* [11]. In this case the simulations have to be performed on a finite system with $L\zeta_c$ particles. Since $L\zeta_c$ is not in general an integer number we have used numbers of particles n and lattice sizes L such that the ratio n/L is as close as possible to ζ_c. Again, although the values of L are not large, a reasonable collapse of the data is obtained.

V. CONCLUSION

We have introduced a procedure that allowed us to get the stationary distribution of models with absorbing states. Two one-dimensional models were analyzed: the contact process and a sandpile model with height restriction. The method presented here permitted the study of the subcritical regime since the absorbing state has been suppressed. By means of a finite-size scaling analysis we obtained the critical behavior. The perturbation introduced is sufficiently small in the sense that the properties, in the thermodynamic limit, will be the same as in the unperturbed model.

ACKNOWLEDGMENT

We acknowledge CNPq for financial support.

[1] T. M. Liggett, *Interacting Particle Systems* (Springer-Verlag, New York, 1985).
[2] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge, U.K., 1999).
[3] H. Hinrichsen, Adv. Phys. **49**, 815 (2000).
[4] S. S. Manna, J. Phys. A **24**, L363 (1991).
[5] I. Jensen and R. Dickman, Phys. Rev. E **48**, 1710 (1993).
[6] M. Rossi, R. Pastor-Satorras, and A. Vespignani, Phys. Rev. Lett. **85**, 1803 (2000).
[7] R. Dickman, M. A. Muñoz, A. Vespignani, and S. Zapperi, Braz. J. Phys. **30**, 27 (2000).
[8] R. Dickman, M. Alava, M. A. Muñoz, J. Peltola, A. Vespignani, and S. Zapperi, Phys. Rev. E **64**, 056104 (2001).
[9] S. Lubeck, Phys. Rev. E **64**, 016123 (2001).
[10] S. Lubeck, Phys. Rev. E **66**, 046114 (2002).
[11] R. Dickman, T. Tomé, and M. J. de Oliveira, Phys. Rev. E **66**, 016111 (2002).
[12] M. M. de Oliveira and R. Dickman, Physica A **343**, 525 (2004).
[13] R. M. Ziff and B. J. Brosilow, Phys. Rev. A **46**, 4630 (1992).
[14] T. Tomé and M. J. de Oliveira, Phys. Rev. Lett. **86**, 5643 (2001).
[15] T. Vicsek, *Fractal Growth Phenomena*, 2nd ed. (World Scientific, Singapore, 1992).